Class 10/13/2016

Note Title 9/30/2016

Theorem:

Let f be analytic inside and on a simple contour C taken with positive orientation. If z_0 is a point inside C then:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \quad n = 0,1,2\ldots$$

Example 1:

Let $P(z) = a_0 + a_1 z + \ldots + a_m z^m$,

We can compute the integral

$$\oint_{|z|=1} \frac{P(z)}{z^{m+1}} \quad \text{in two ways}$$

1. $$\oint_{|z|=1} \frac{P(z)}{z^{m+1}} = \oint_{|z|=1} \frac{a_0}{z^m}\,dz + \oint \frac{a_1}{z^{m-1}}\,dz + \ldots + \oint \frac{a_m}{z}\,dz$$

we already know that

$$\oint z^m\,dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m = -1 \end{cases}$$

Thus, we conclude that

$$\oint \frac{P(z)}{z^{m+1}}\,dz = \boxed{2\pi i\, a_m.}$$

2. On the other hand theorem gives:

$$\oint \frac{P(z)}{z^{m+1}}\,dz = \frac{2\pi i}{m!} P^{(m)}(0)$$

$$P^{(m)}(0) = m\cdot(m-1)\cdot(m-2)\cdots 1\cdot a_m = m!\,a_m$$

$$\Rightarrow \oint \frac{P(z)}{z^{m+1}}\,dz = \boxed{2\pi i\, a_m}$$

So we obtain the same answer.

Example 2:

$$I = \oint \frac{\exp(2z)\,dz}{z^4}$$ denote $f(z) = \exp(2z)$ then:

$$I = \oint \frac{f(z)\,dz}{(z-0)^{3+1}} \overset{\text{Theorem}}{=\!=} \frac{2\pi i}{3!} f'''(0)$$

$$f'''(z) = 8\exp(2z) \Rightarrow f'''(0) = 8$$

$$\Rightarrow \boxed{\frac{8\pi i}{3}}$$

Example 3

$$\oint_{|z|=1} \frac{\cos z}{z^5}\,dz = \oint_{|z|=1} \frac{\cos z}{(z-0)^{4+1}}\,dz = \frac{2\pi i}{4!}\cos''''(0)$$
$$= \boxed{\frac{\pi i}{12}}$$

Example 4

Let C - be closed positively oriented contour

let
$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\,ds$$

Show that $g(z) = 6\pi i z$ if z is inside

$g(z) = 0$ else.

Solution: if z not in C $\Rightarrow$ function is analitic $\Rightarrow$ $\boxed{0}$

if z is in C, then:

$$g(z) = \frac{2\pi i}{2!}\left(s^3 + 2s\right)''\Big|_{s=z} = \pi i\,(6 \cdot s)\Big|_{s=z} = \boxed{6\pi i}$$

Example 5

$$\oint_{|z|=1} \frac{e^{\alpha z}}{z^{m+1}}\,dz = \frac{2\pi i}{m!}\left(e^{\alpha z}\right)^{(m)}\Big|_{z=0} = \boxed{\frac{2\pi i \alpha^m}{m!}}$$

Example 6

$$\oint_{|z-x_0|=1} \frac{\tan(z/2)}{(z-x_0)^2}\,dz = 2\pi i\,\left(\tan(z/2)\right)'\Big|_{z=x_0} =$$

$$= \boxed{\pi i \sec(x_0/2)}$$